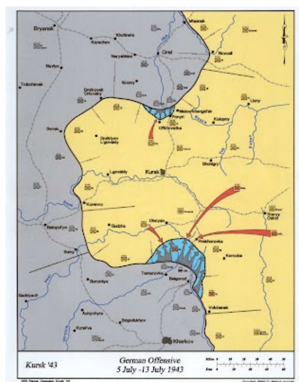
# Fitting Lanchester Equations to the Battle of Kursk and Other Time-Phased Battle Data

by

**Tom Lucas**, Naval Postgraduate School (NPS)

**First Historical Analysis Annual Conference (HAAC)**

**27-29 September 2022**

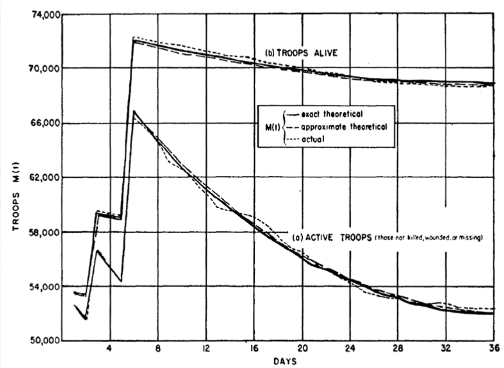




FIG. 1: $M(t)$, number of United States troops in action.

$$\frac{dx}{dt} = -ay \quad and \quad \frac{dy}{dt} = -bx$$

$$\frac{dx}{dt} = -axy \quad \text{and} \quad \frac{dy}{dt} = -bxy$$

# This Research is based on …

- **Stymfal, Matthew, Lieutenant Commander, U.S. Navy, "Revisiting Engle's Verification of Lanchester's Square Law Using Battle of Iwo Jima Data," M.S. in Operations Research, September 2022.**

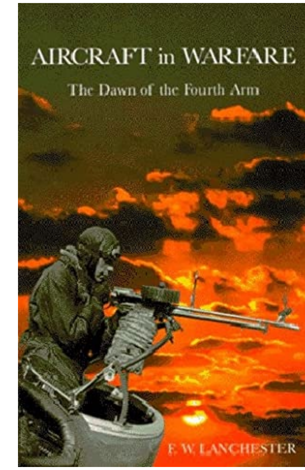
- **Dinges, John, Captain, U.S. Army, "Exploring the Validation of Lanchester Equations for the Battle of Kursk," M.S. in Operations Research, June 2001.**

- **Turkes, Turker, First Lieutenant, Turkish Army, "Fitting Lanchester and Other Equations to the Battle of Kursk Data," M.S. in Operations Research, March 2000.**

# Outline

- Quick Review of Lanchester Equations
- Overview of the Battle of Kursk
- Some Previous Research on Fitting Lanchester Equations to Time-phased Battle Data
- Lanchester and the Battle of Kursk
- Parting thoughts

# Lanchester's Two Models




- Lanchester's (1914) differential equation attrition models that explore the "**principle of concentration of forces**"
- Lanchester's two models
  - Assuming **homogeneous forces**
- "Modern Warfare" ==> $$\frac{dx}{dt} = -ay \quad and \quad \frac{dy}{dt} = -bx$$
  - "The rate at which a force is depleted is proportional to the size of the enemy and the individual capability of the enemy."
- "Ancient Warfare" ==> $$\frac{dx}{dt} = -axy \quad \text{and} \quad \frac{dy}{dt} = -bxy$$
  - "assume the fire [is] concentrated on a certain area known to be held by the enemy" or "firing … 'into the brown'."
  - "where man is opposed to man."

**Note: For "ancient warfare":** $$\frac{dx}{dy} = E$$

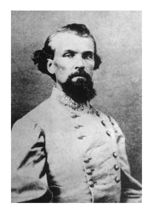
# Lanchester Concluded


- In modern war there is added benefit to **concentration of forces**!
  - *It is an "important truth" that "the fighting strength of a force can be represented by the square of its numerical strength."*
- Mathematical explanation of
  - "Divide and conquer"

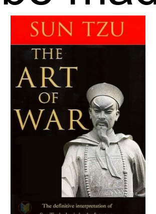
  - "God is on the side of the big battalions"

  - "Superiority in numbers is the most important factor in the result of a combat … the greatest possible number of troops should be brought into action at the decisive point."

  - "Getting there firstest with the mostest"



  - "[T]hough an obstinate fight may be made by a small force, in the end it must be captured by the larger force."




**"The value … that best fits the empirical data … is 1.5" — M. Osipov (1915)**

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# The Battle of Kursk

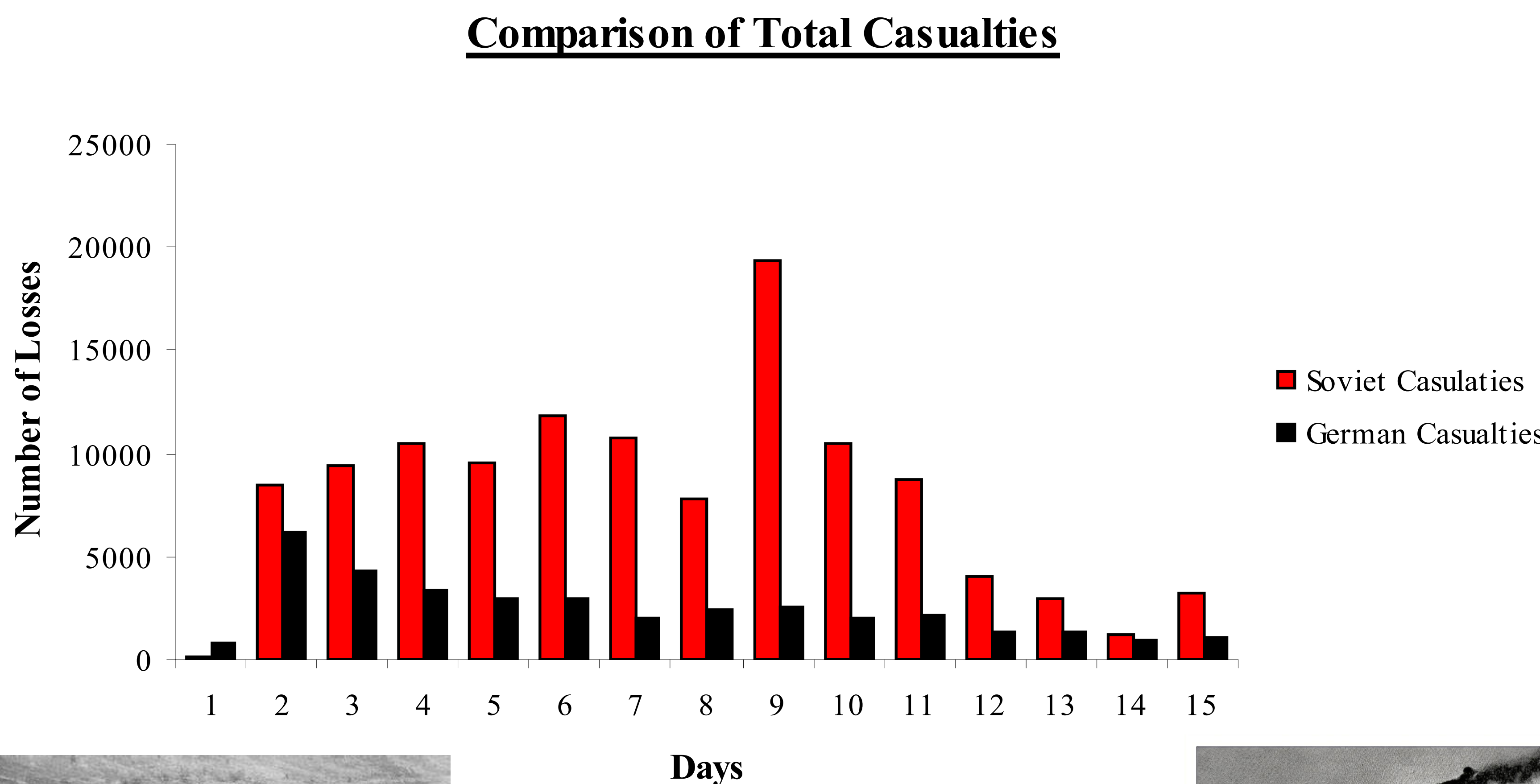
"This attack is of decisive importance. It must succeed, and it must do so rapidly and convincingly. It must secure for us the initiative for this spring and summer. The victory of Kursk must be a blazing torch to the world."
— Hitler, 2 July 1943

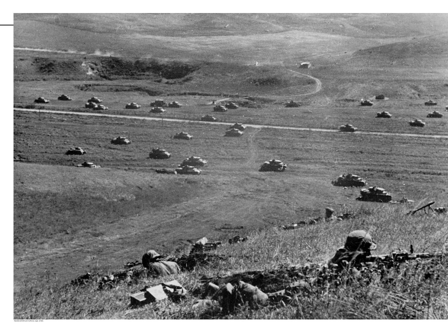
# The Data Sources…

*Kursk Operations Simulation and Validation Exercise – Phase II (KOSAVE II)*, The U.S. Army's Center for Strategy and Force Evaluation Study Report CAA-SR-98-7, September 1998.

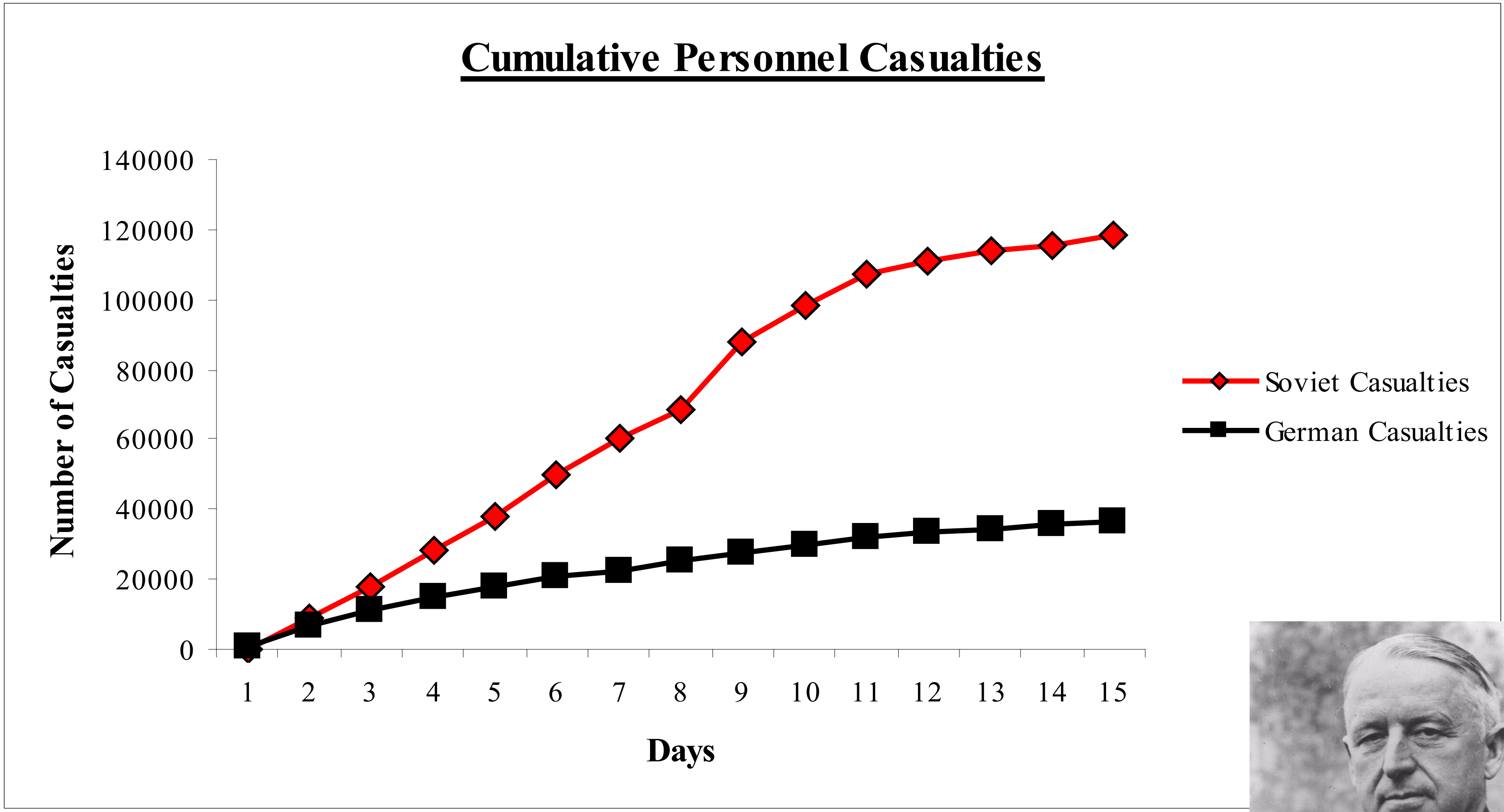
Data Memory Systems Inc., *The Ardennes Campaign Simulation Data Base (ACSBD), Phase II Final Report*, 1989.

# Daily Personnel Casualties



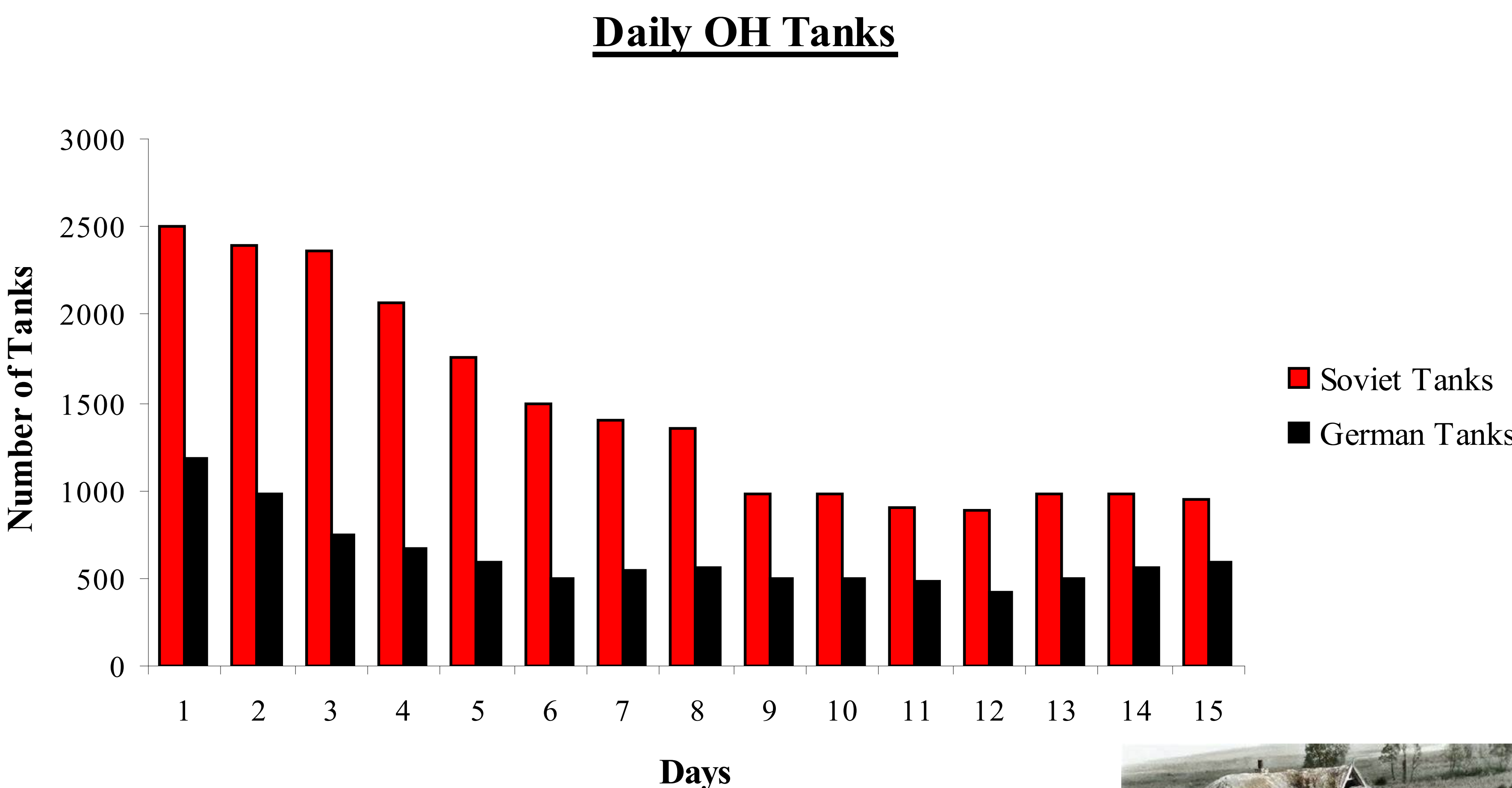



# Personnel Casualties

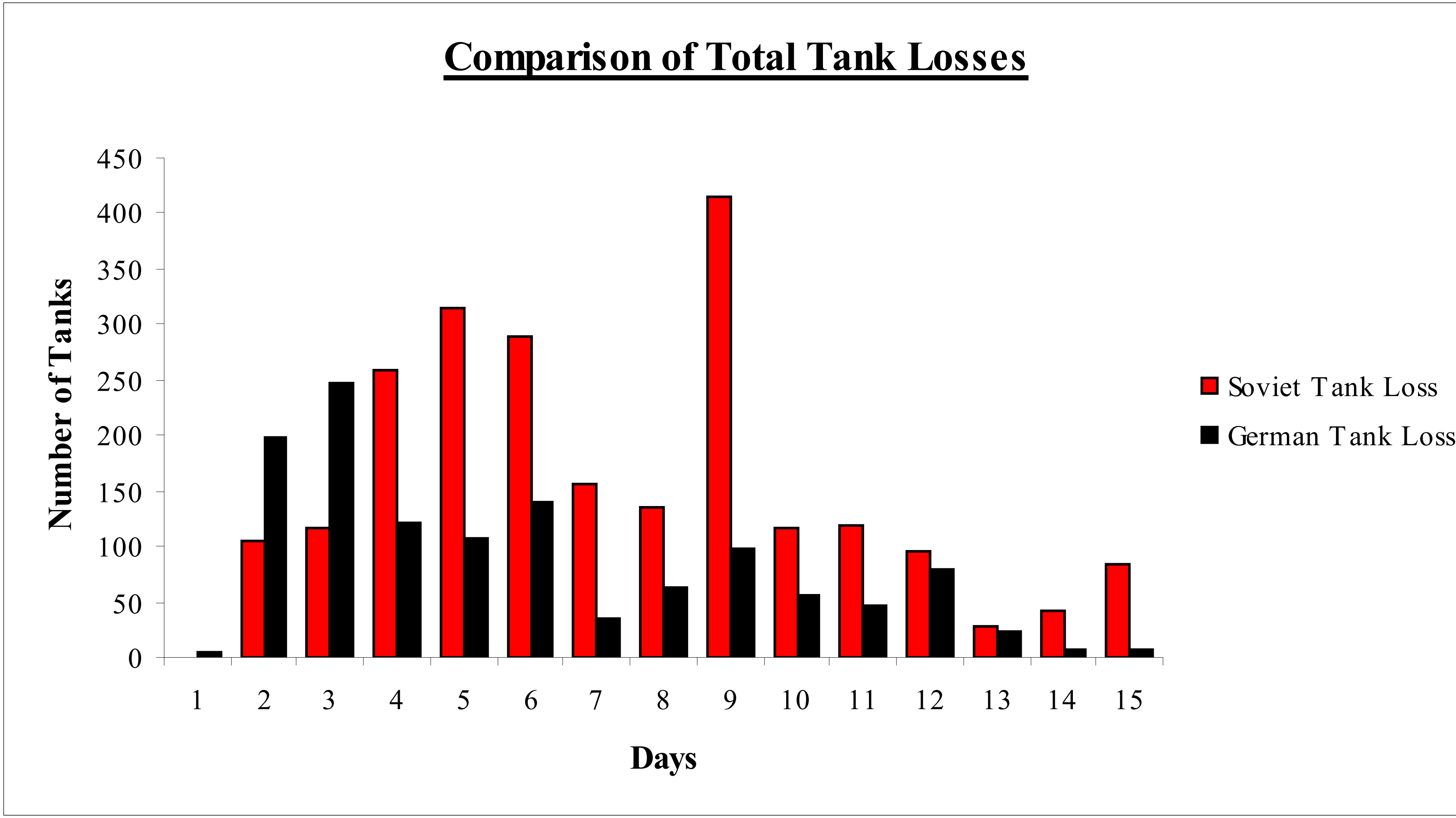



**Field Marshal Erich von Manstein, Commander of Army Group South: "[stopping the offensive] at this moment [is] tantamount to throwing victory away … the last German offensive in the East ended in a fiasco, even though the enemy ... suffered four times their losses."**

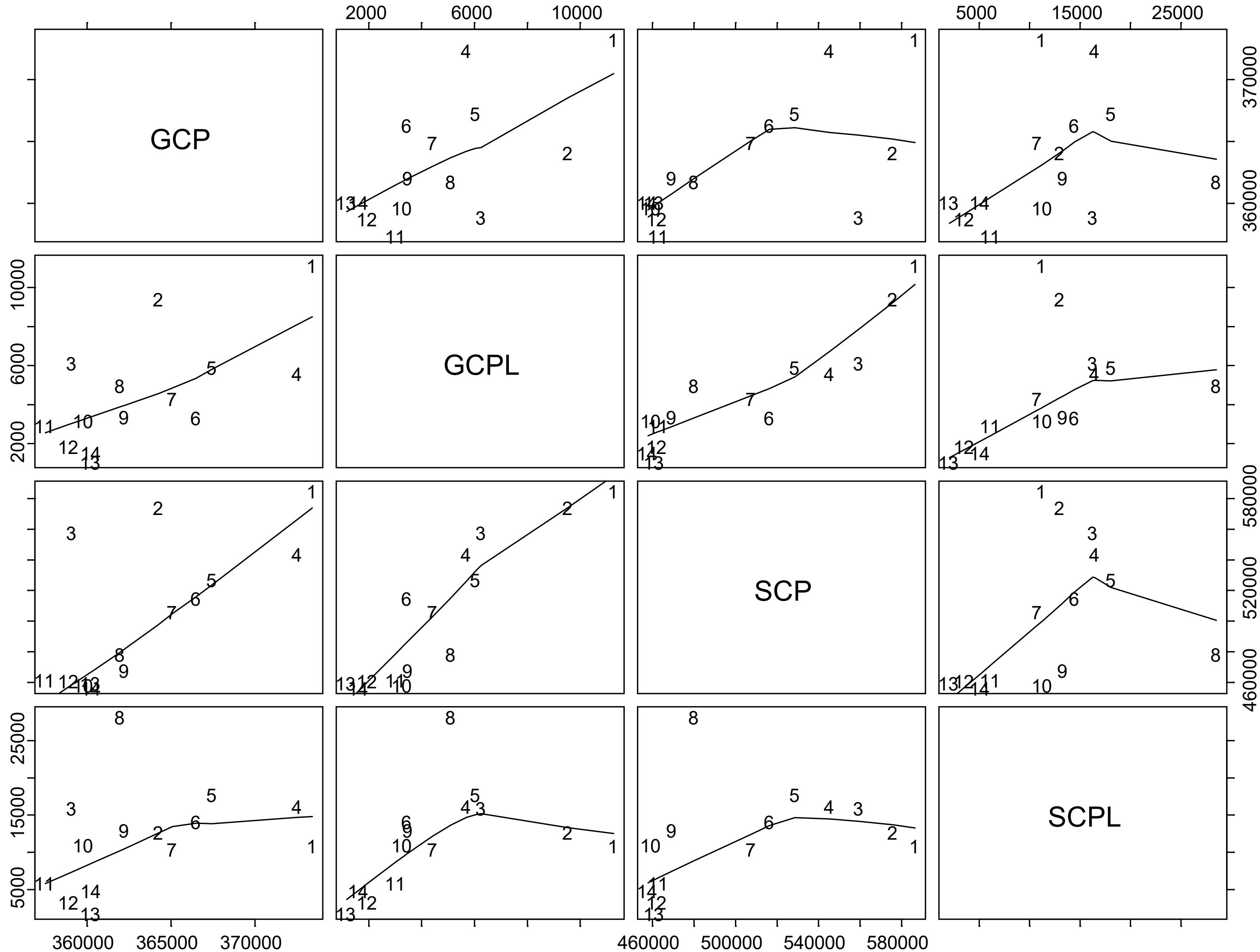
# Daily On-hand Tanks




# Daily Tank Losses



# Pairwise Scatter Plots on Combat Power and Combat Power Losses




**Using Bracken's (1995) weights for combat power: pers = 1, APC = 5, Tanks = 20, Arty = 40**

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# Lanchester's Square Law Fit to the Battle of Iwo Jima

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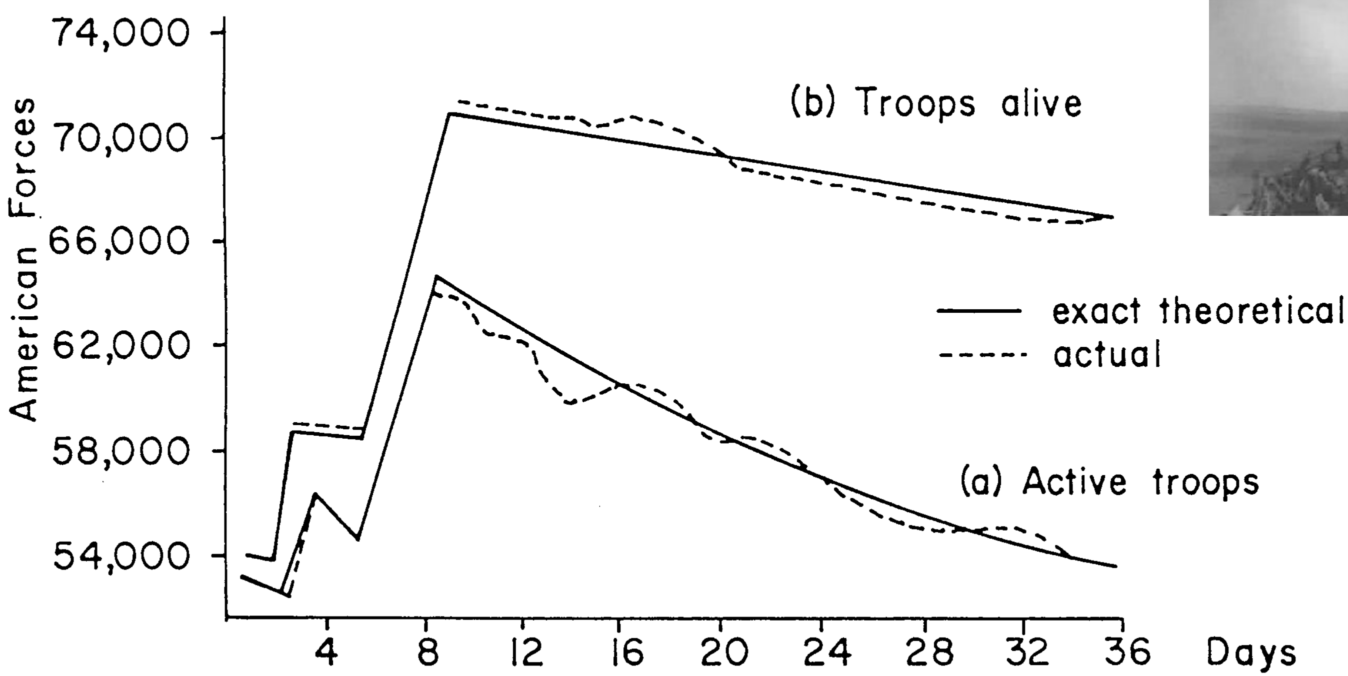



Figure 8.6. Comparison of Actual Troop Strength with Theoretical for the Battle of Iwo Jima (Adapted from Engel [3])

**Active Troops are defined by Engel as "those not killed, wounded or missing"**

**Engel's fit had the Japanese attrition coefficient (0.0544 per day) as 5.1 times that of the Marines (0.0106 per day)**

**"In the analysis of the capture of Iwo Jima, the equations were found to be applicable. The value of such analysis increases when repeated often enough to permit general conclusions to be drawn." – Engle (1954)**

**Follow-on research is not as supportive…**

$$\frac{dx}{dt} = -ax^q y^p$$

$$\frac{dy}{dt} = -bx^p y^q$$

# Summary of models, extensions, and fitness measure (Stymfal 2022)

| Model | p | q | a | b | $R^2$ |
|---|---|---|---|---|---|
| Engel | 1 | 0 | 0.0544 | 0.0106 | 0.9937 |
| Square | 1 | 0 | 0.0533 | 0.0105 | 0.9944 |
| Linear | 1 | 1 | $2.31 \times 10^{-6}$ | $2.28 \times 10^{-6}$ | 0.9029* |
| Logarithmic | 0 | 1 | 0.0108 | 0.5160 | 0.9414* |
| Bracken | **1.05** | **0** | 0.0331 | 0.0061 | 0.9946 |

***Fit to Japanese end strength of 200**

# Hartley and Hembold (1995) on Inchon-Seoul

**The method: Linear regression to assess Lanchester's square law**

**Key Findings:**

- **"Any square law effects are largely masked by other factors"**
- **The data better fit a set of three separate battles (one distinct battle every six or seven days)**
- **"Unless we are able to procure data…with many periodic casualty figures for both sides, we will not be able to validate the homogeneous square law (or any other attrition law)"**

# Bracken's (1995) Work on Ardennes

**The model:**

$$\dot{B} = a(d \text{ or } 1/d)R^p B^q$$
$$\dot{R} = b(1/d \text{ or } d)B^p R^q$$

**CP weights:** **personnel = 1, APC = 5, Tanks = 20, Arty = 40**

**The MOE and method:** **Min SSR between actual and estimated losses by a 5-dimensional constrained grid search over the first 10 days of the battle.**

# Bracken's Results

| | COMBAT MANPOWER | SUPPORT MANPOWER | PARAMETER *d* |
|---|---|---|---|
| MODEL1 | X | | X |
| MODEL2 | X | X | X |
| MODEL3 | X | | |
| MODEL4 | X | X | |

| **model** | ***a*** | ***b*** | ***p*** | ***q*** | ***d*** |
|---|---|---|---|---|---|
| Model 1 | 8.0E-9 | 1.0E-8 | 1.0 | 1.0 | 1.25 |
| Model 2 | 8.0E-9 | 8.0E-9 | 0.8 | 1.2 | 1.25 |
| Model 3 | 8.0E-9 | 1.0E-8 | 1.3 | 0.7 | - |
| Model 4 | 8.0E-9 | 8.0E-9 | 1.2 | 0.8 | - |

**"[T]he Lanchester linear model fits the Ardennes campaign data in all four cases"**

# Fricker's (1998) Work on Ardennes

**The same model:**

$$\dot{B} = a(d \text{ or } 1/d)R^p B^q$$
$$\dot{R} = b(1/d \text{ or } d)B^p R^q$$

**Changes include: whole battle data, reformatted data, and added air sorties**

**The MOE and method: Min SSR between actual and estimated losses by linear regression on logarithmically transformed equations (unconstrainted *p* and *q*).**

# Fricker's Findings

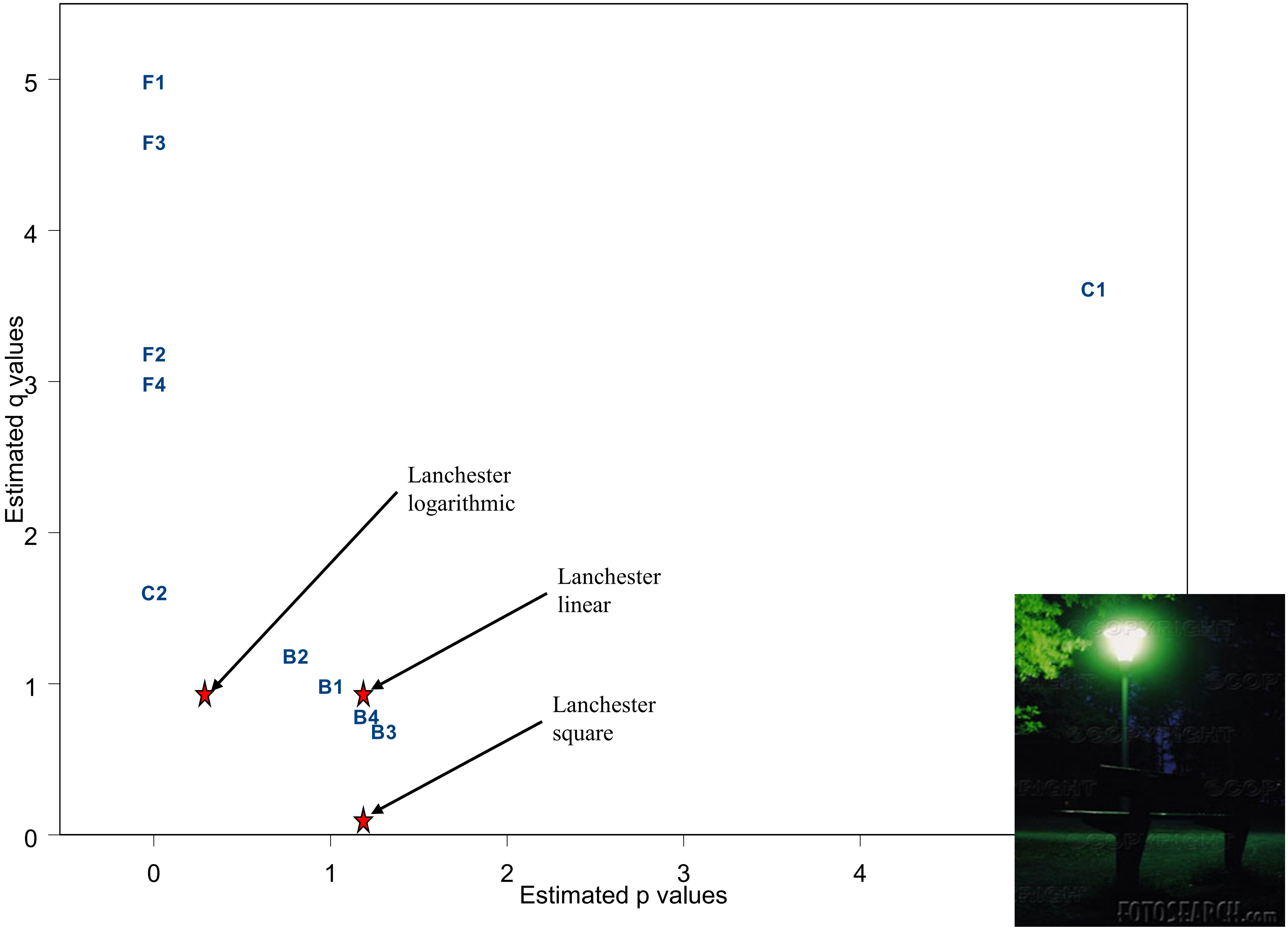
| **Name of the model** | ***a*** | ***b*** | ***p*** | ***q*** | ***d*** |
|---|---|---|---|---|---|
| Combat manpower w/o sortie | 4.7E-27 | 3.1E-26 | 0.0 | 5.0 | 0.8093 |
| Total manpower w/o sortie | 1.7E-16 | 8.0E-16 | 0.0 | 3.2 | 0.824 |
| Combat manpower With sortie | 2.7E-24 | 1.6E-23 | 0.0 | 4.6 | 0.7971 |
| Total manpower with sortie | 1.3E-15 | 5.6E-15 | 0.0 | 3.0 | 0.8197 |

# Clemens’s (1997) Findings on Kursk

## Same model as Bracken, days 2 –15 of Kursk data

| Name of the model | *a* | *b* | *p* | *q* | *d* |
|---|---|---|---|---|---|
| Clemens Linear Regression | 6.92E-49 | 6.94E-48 | 5.3157 | 3.6339 | - |
| Clemens Newton-Raphson | 3.73E-6 | 5.91E-6 | 0.0 | 1.6178 | - |

# Plot of Cumulative Findings for *p* and *q*



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# Basic Lanchester Models (no *d*)

| **Name of the model** | ***a*** | ***b*** | ***p*** | ***q*** | $R^2$ |
|---|---|---|---|---|---|
| Lanchester Linear model | 6.68E-8 | 2.68E-8 | 1.0 | 1.0 | 0.1290 |
| Lanchester Square model | 0.0335 | 0.0098 | 1.0 | 0 | 0.0521 |
| Lanchester Logarithmic model | 0.0243 | 0.0131 | 0 | 1.0 | 0.0831 |
| Morse Kimball Equations | a=-0.041 | $\alpha_1$=0.053 | b=0.060 | $\alpha_2$=-0.07 | 0.2297 |

**Co-linearity is a serious issue, especially with Morse-Kimball**

# Basic Lanchester Models (with *d*)

**Table 5.** Lanchester law fits for the battle of Kursk, with the tactical parameter $d$, for both Bracken's [2] weights and combat manpower.[a]

| Lanchester law | Weights | $p$ | $q$ | $d$ | $R^2$ |
|---|---|---|---|---|---|
| Square | Bracken's | 1 | 0 | 1.09 | .081 |
| Linear | Bracken's | 1 | 1 | 1.02 | .131 |
| Logarithmic | Bracken's | 0 | 1 | 1.02 | .085 |
| Optimum fit | Bracken's | 5.87 | 1.01 | 1.03 | .238 |
| Square | Manpower | 1 | 0 | 1.11 | .074 |
| Linear | Manpower | 1 | 1 | 1.04 | .116 |
| Logarithmic | Manpower | 0 | 1 | 1.04 | .086 |
| Optimum fit | Manpower | 7.74 | 3.41 | .86 | .234 |

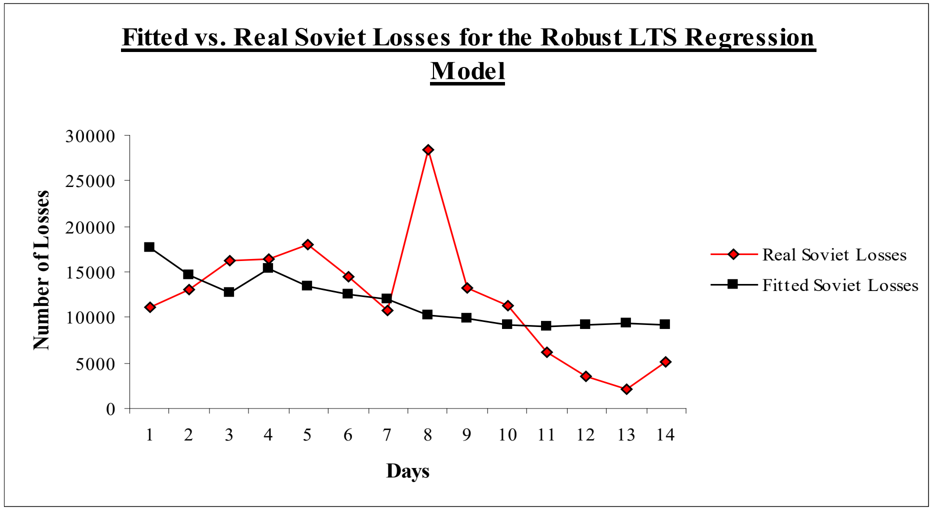
[a] In both cases, the linear law is the best fitting of the basic laws; however, the linear law is not close to the optimal fit.

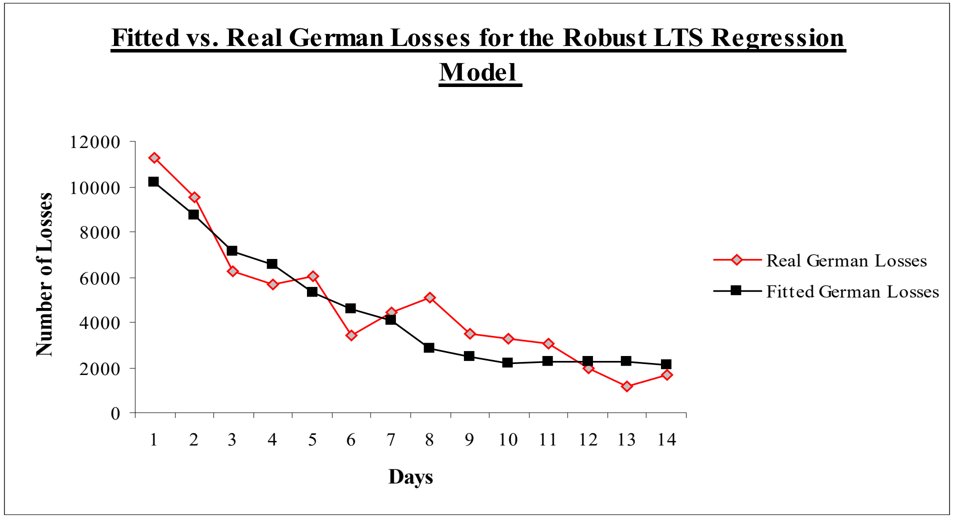
# Best Fitting (by LTS Regression on Logarithmically Transformed Data)

$$\dot{B} = 2.27 \times 10^{-40} R^{6.0843} B^{1.7312}$$

$$\dot{R} = 1.84 \times 10^{-41} B^{6.0843} R^{1.7312}$$

**$R^2 = 0.2262$**




**Note: adding in air-sortie data resulted in a poorer fit!**

# Different Force Strength Weights

| **Name of the model** | ***a*** | ***b*** | ***p*** | ***q*** | $R^2$ |
|---|---|---|---|---|---|
| Weight Comb.1 (1,5,40,20) | 7.26E-35 | 5.53E-36 | 5.5312 | 1.3268 | 0.1514 |
| Weight Comb.2 (1,5,20,15) | 7.85E-36 | 4.75E-37 | 5.8613 | 1.1899 | 0.2072 |
| Weight Comb.3 (1,5,40,30) | 1.46E-35 | 9.33E-37 | 5.9619 | 1.0159 | 0.1637 |
| Weight Comb.4 (1, 5, 20,30) | 5.05E-35 | 3.51E-36 | 5.6294 | 1.2631 | 0.1873 |

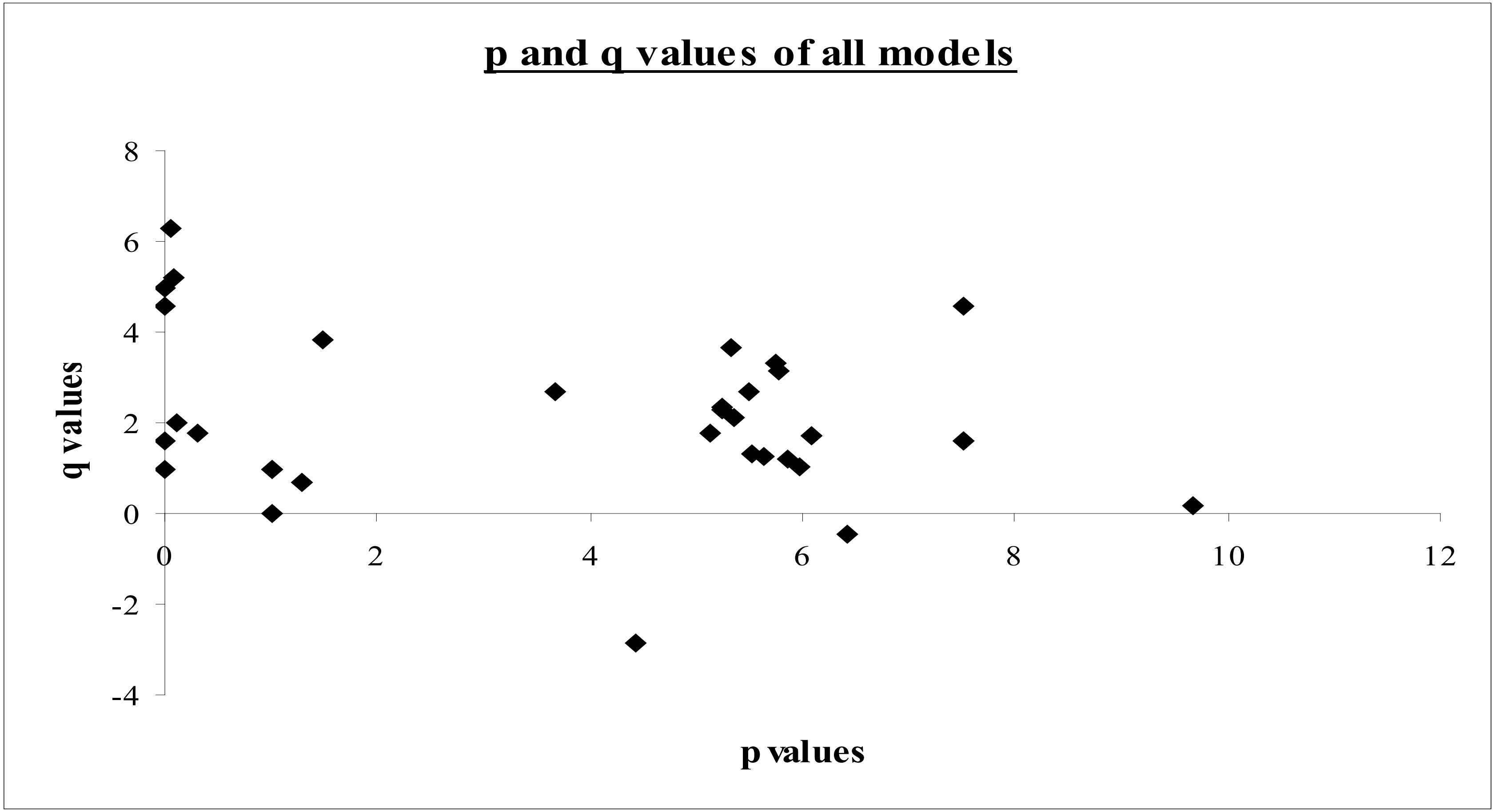
# Change Points: Four Different *d*'s

| **Name of the model** | ***a*** | ***b*** | ***p*** | ***q*** | ***d*** | ***SSR*** | **$R^2$** |
|---|---|---|---|---|---|---|---|
| Campaign in four Parts | 1.88E-47 | 1.07E-48 | 7.5068 | 1.5793 | 4 periods *d*=0.91,1.24, 0.32,1.17 | 1.69E+8 | .764 |
| Leave out Day 8, no d | 1.85E-51 | 3.56E-53 | 9.6853 | 0.1458 | - | 1.90E+8 | 0.5658* |

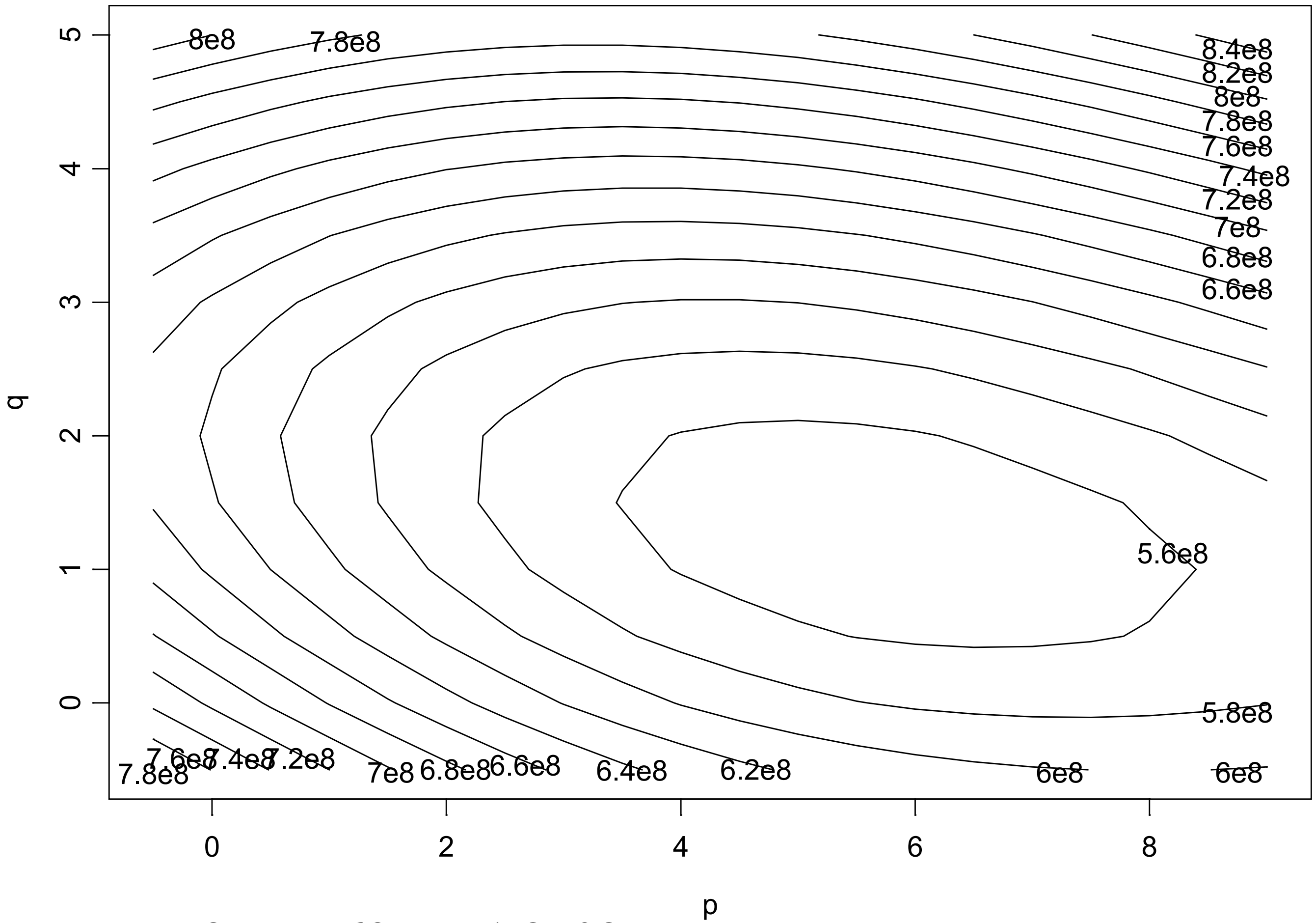
- **8 parameters with 14 days of data**
- *** = different SST**

**A *posteriori* fitting to natural battle phases**

# All of Turkes' Fits



# A Better Approach: The Kursk Surface



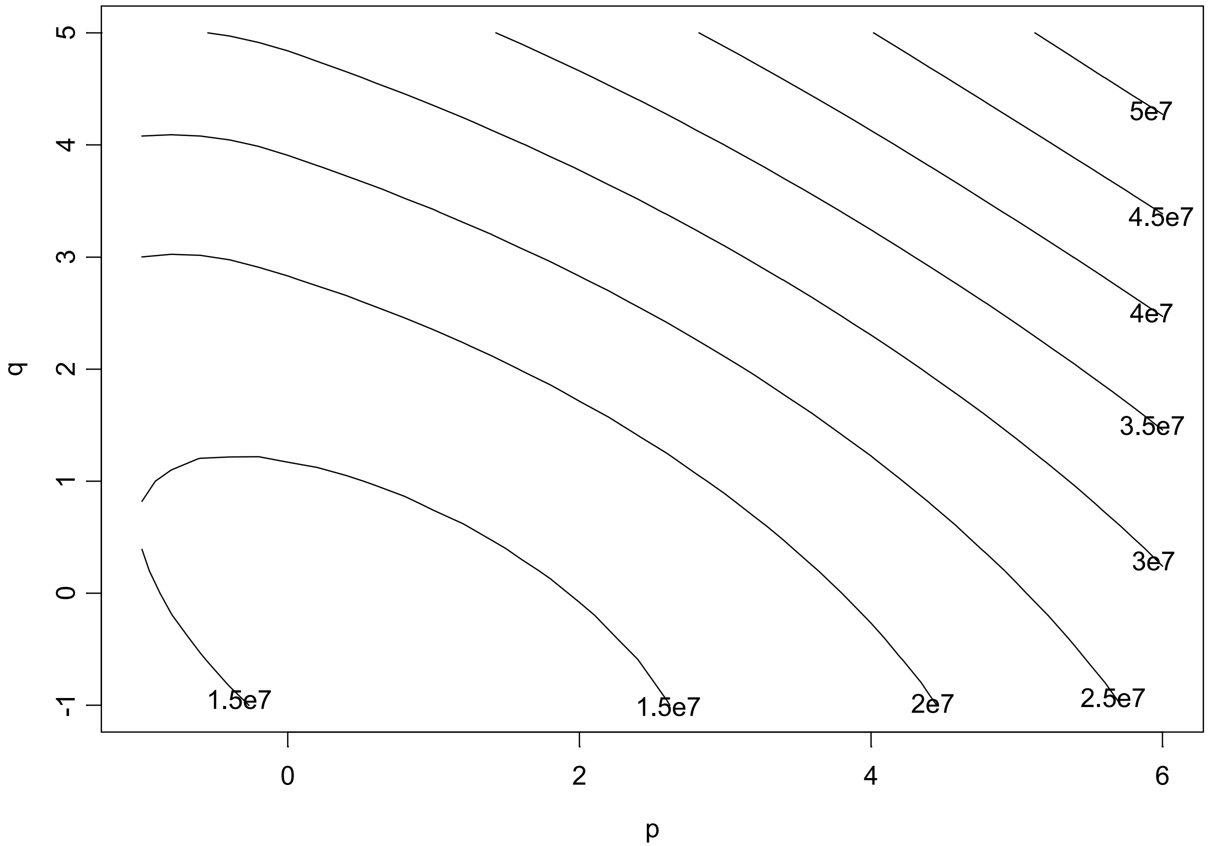

$$\dot{B} = 1.4658 \times 10^{-35} R^{5.6957} B^{1.2702}$$

$$\dot{R} = 1.2014 \times 10^{-36} B^{5.6957} R^{1.2702}$$

**$R^2 = 0.237$**

*** With *d* (= 1.028) $R^2 = 0.238$**

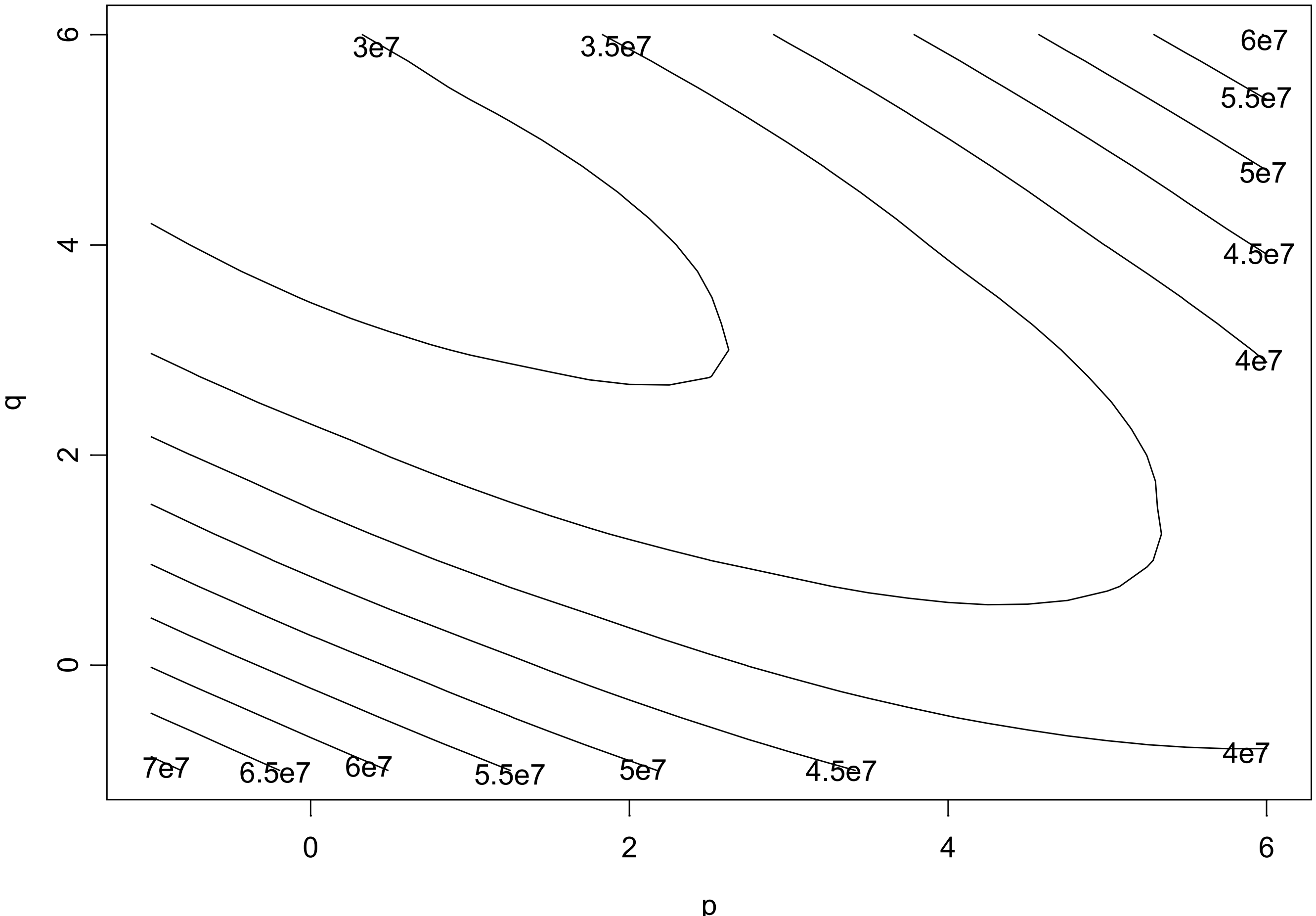
# Ardennes 10-Day Surface




$R^2 = 0.380$

- **Square law fits best**
- **p* = 0.9, q* = -0.6, d* = 1.125**

# Fricker's 32-day Surface




**$R^2$ = 0.500**

- **Logarithmic law fits best**
- **p* = -0.2, q* = 5.0, d* = 1.23**

# Turkes' Conclusions

- Constant attrition homogeneous Lanchester equations do not fit the Kursk data well
  - Linear best of the basic
- Response surface is fairly flat over broad regions (with optimums far from the basic models)
  - It doesn't matter much which Lanchester law you use so long as the coefficients are reasonable
- Kursk and Ardennes give very different best fitting models/surfaces
- Change points (i.e., accounting for varying battle intensities and conditions) dramatically improve fit

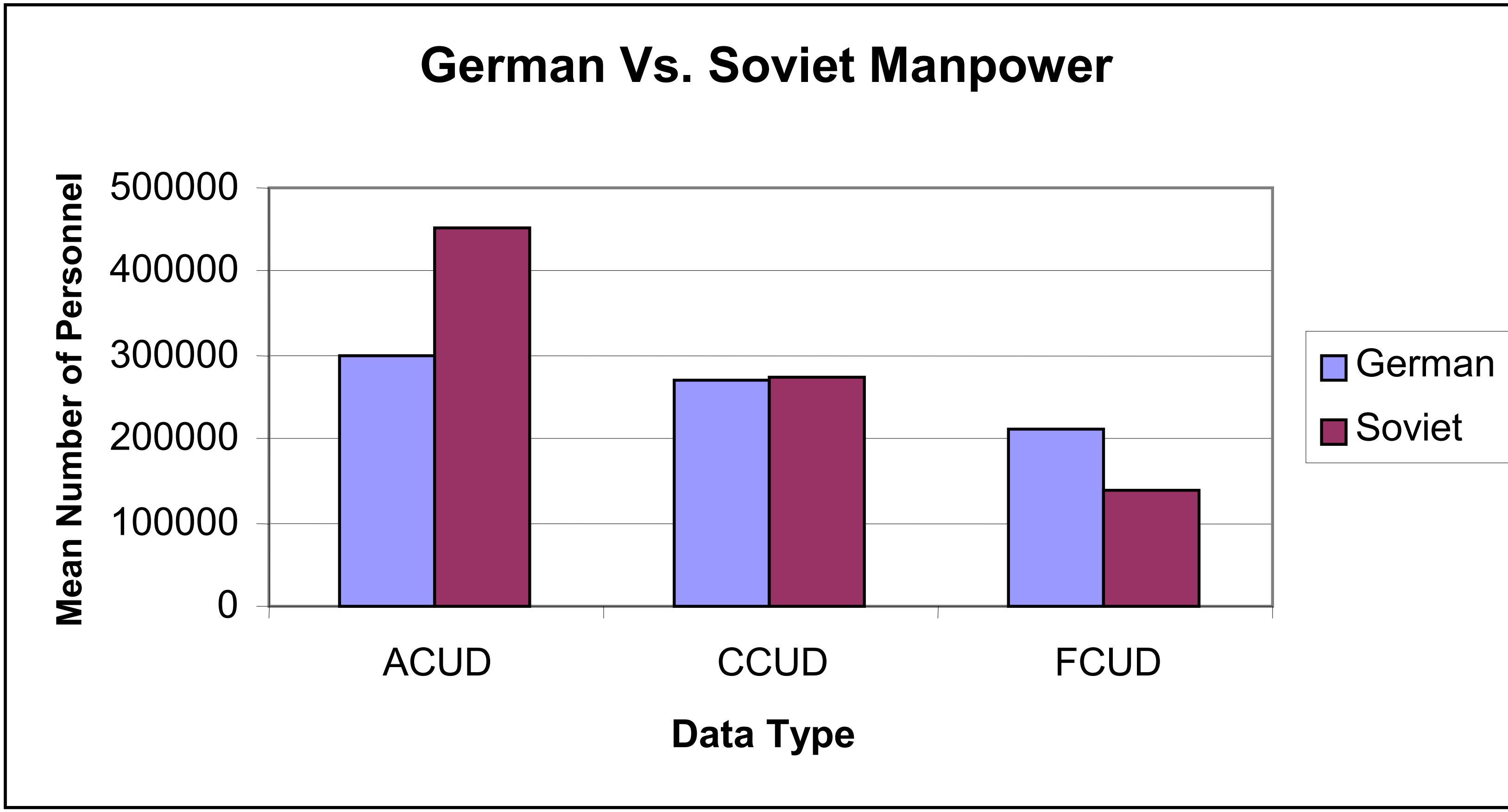
**Building on Turkes and Gozel … Captain John Dinges' (2001) findings on Kursk Database using:**

**(1) All combat unit data (ACUD);**

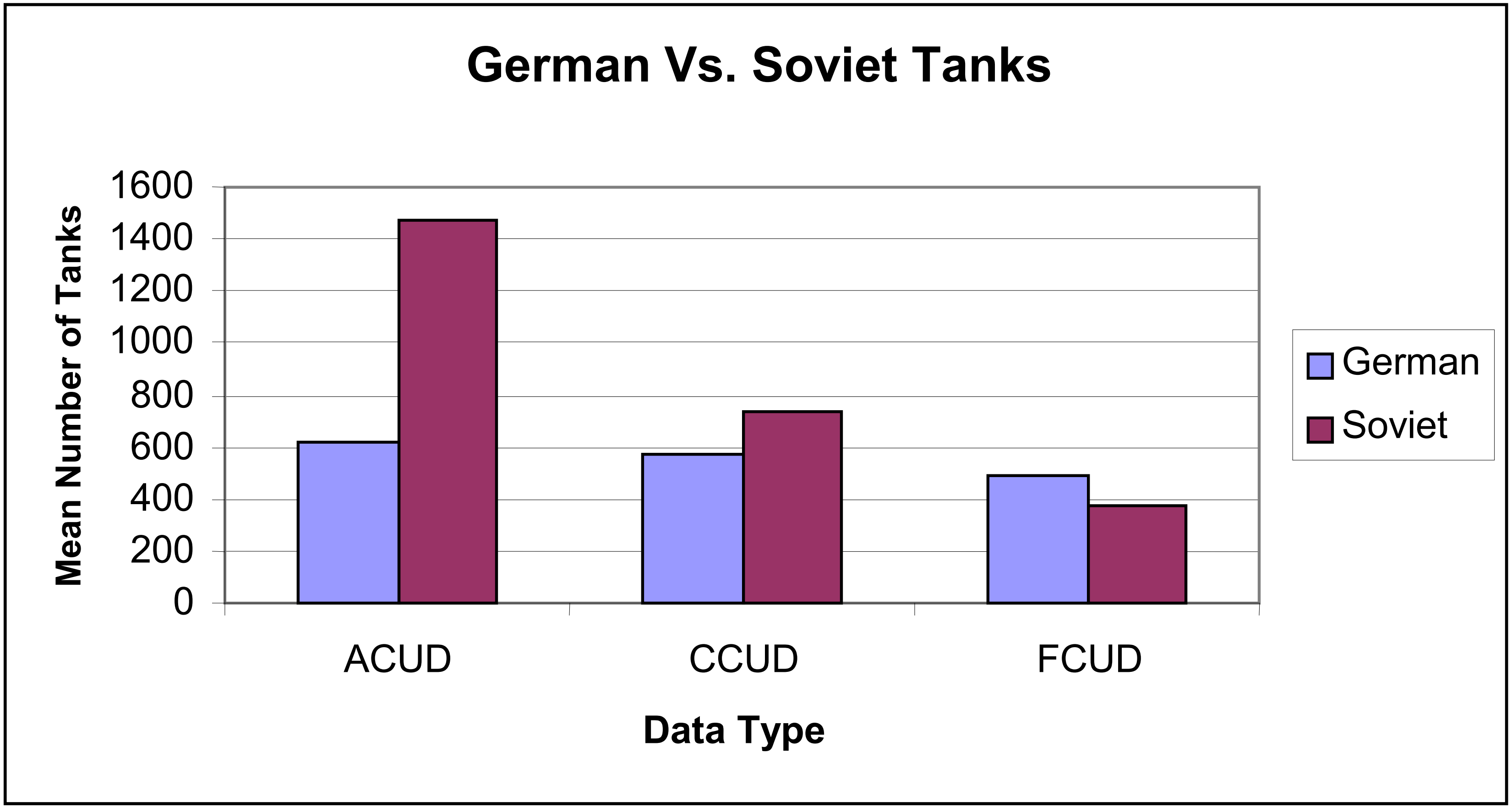
**(2) Combat unit data for those units that are within contact (CCUD); and**

**(3) combat unit data for only those units that are actually fighting (FCUD).**

# Manpower Averages as a Function of Unit Status



# Tank Averages as a Function of Unit Status



# The Basic Models (manpower data)

**Table 3.** Best-fitting basic Lanchester law fits for the three combat status data sets

| Data Set | Lanchester Law | $\hat{a}$ | $\hat{b}$ | $R^2$ |
|---|---|---|---|---|
| All Combat Units (ACUD) | Square | .0284 | .0058 | .034 |
| | Linear | $6.35 \times 10^{-8}$ | $1.96 \times 10^{-8}$ | .110 |
| | Logarithmic | .0190 | .0086 | .079 |
| Contact Combat Units (CCUD) | Square | .0296 | .0081 | .014 |
| | Linear | $1.03 \times 10^{-7}$ | $2.97 \times 10^{-8}$ | .064 |
| | Logarithmic | .0279 | .0092 | .068 |
| Fighting Units (FCUD) | Square | .0333 | .0137 | .298 |
| | Linear | $2.19 \times 10^{-7}$ | $5.89 \times 10^{-8}$ | .622 |
| | Logarithmic | .0481 | .0106 | .535 |

**Key Takeaways**

**(1) FCUD data provide the best fits, by far**

**(2) Linear > Logarithmic >> Square**

# Contour of *p* and *q* fits for FCUD manpower data

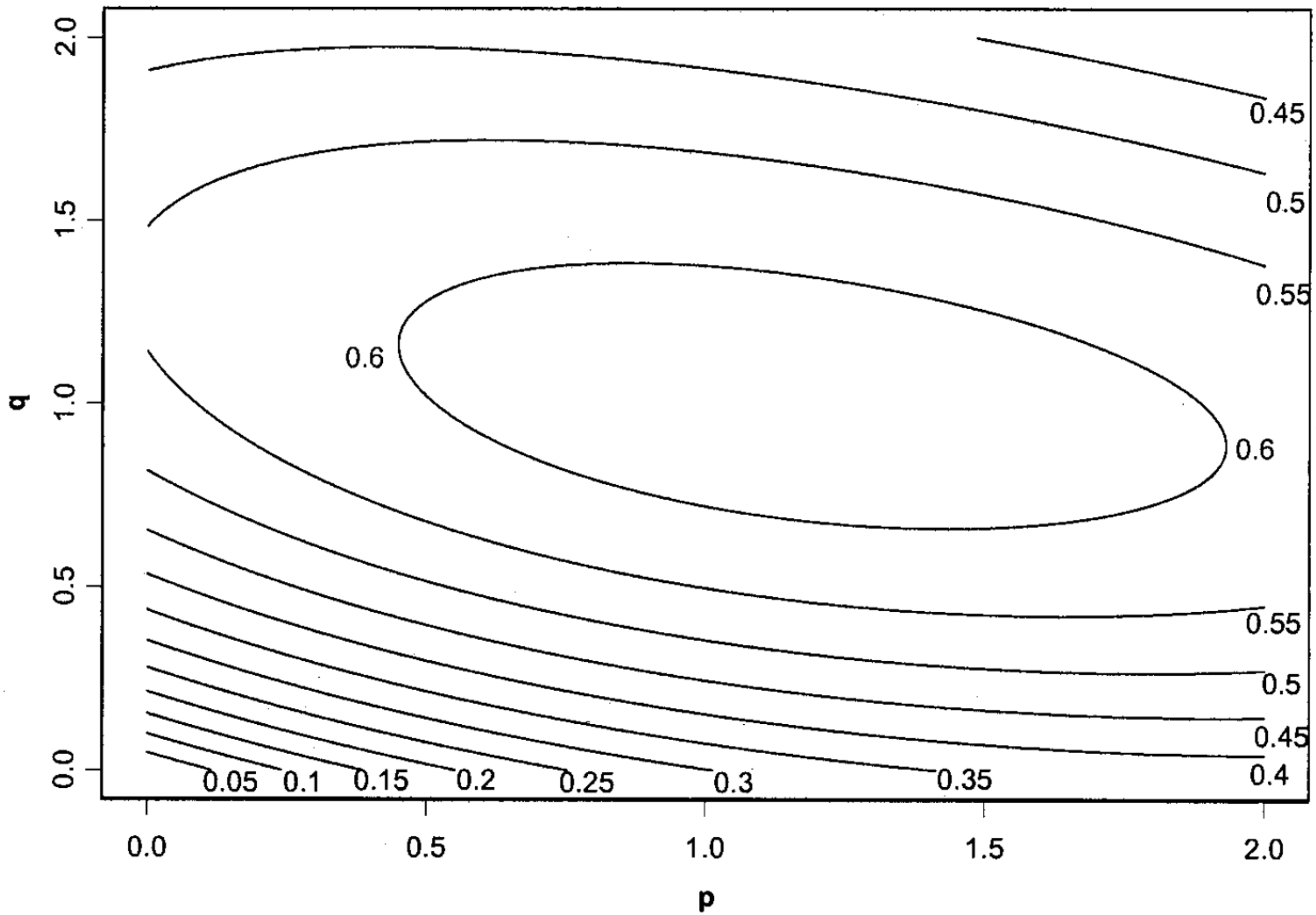



**Figure 1.** Contour plot of the maximum $R^2$ as a function of exponent parameters $p$ and $q$.

# Tracing the Battle for ACUD

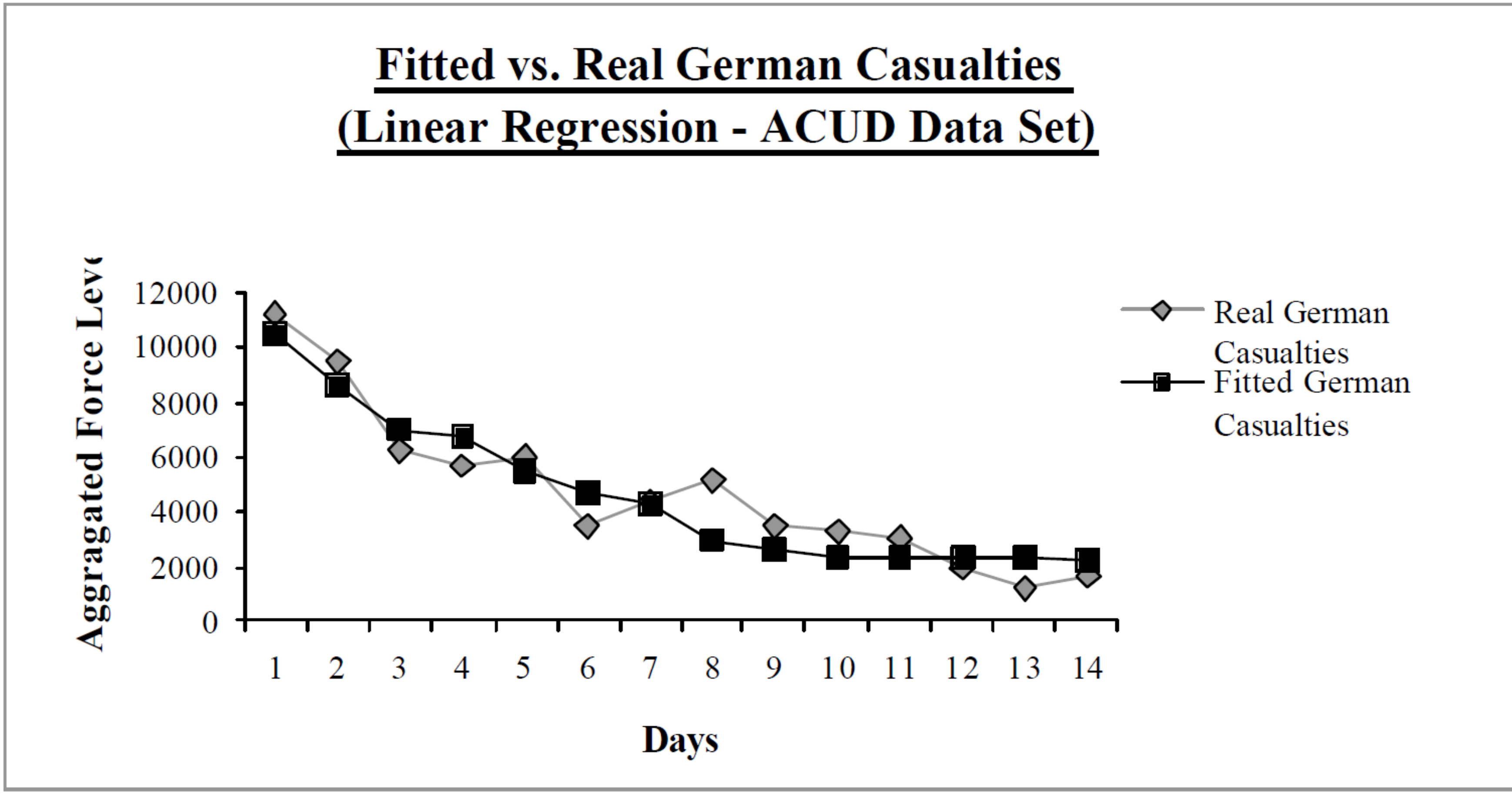



Figure III.1. Fitted versus real German casualties for ACUD data set.

# Tracing the Battle for ACUD

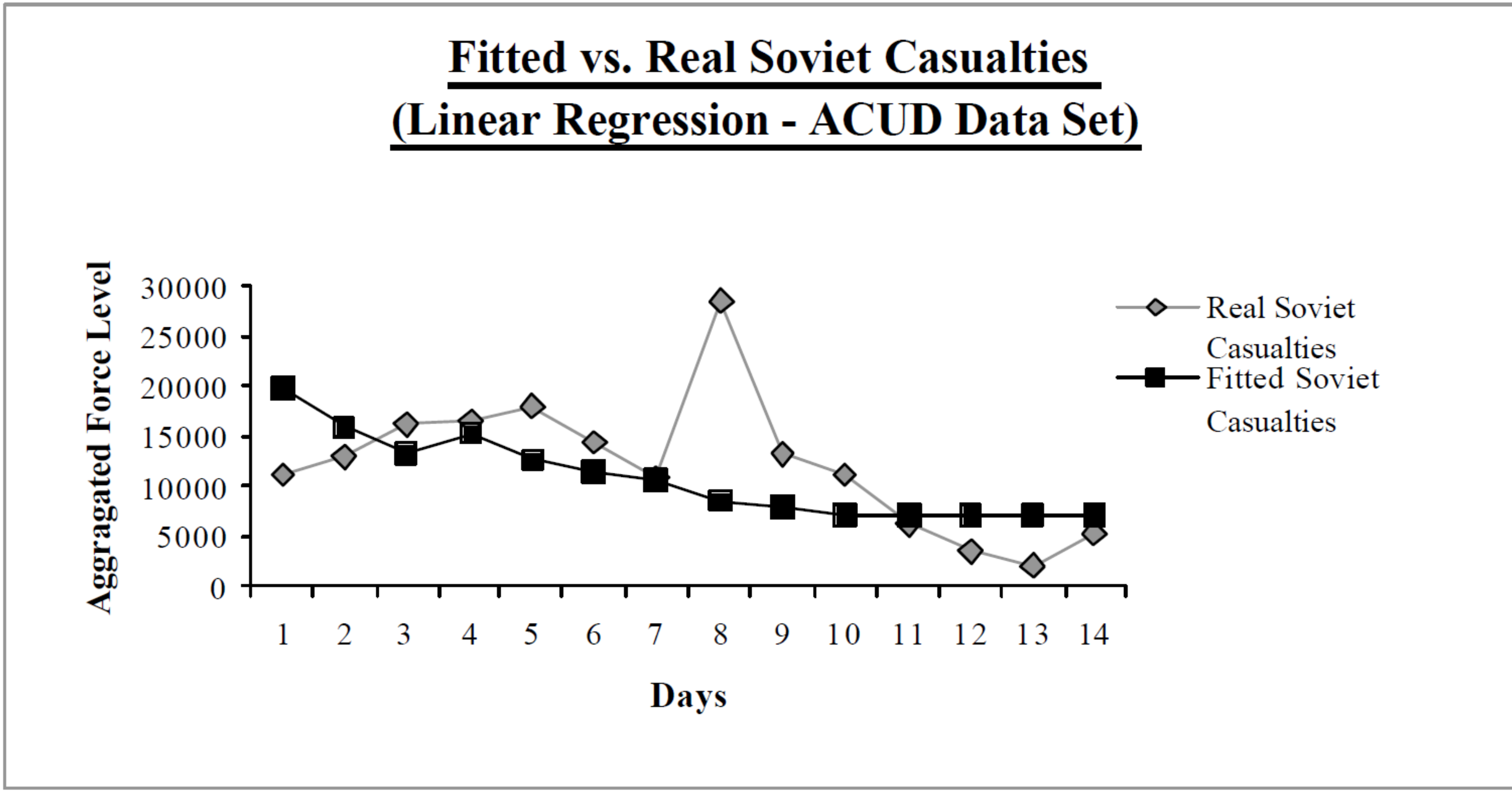



Figure III.2. Fitted versus real Soviet casualties for ACUD data set. Notice the large outlier on day eight. This outlier seems to directly contribute to a higher *SSR* and lower $R^2$.

# Tracing the Battle for FCUD

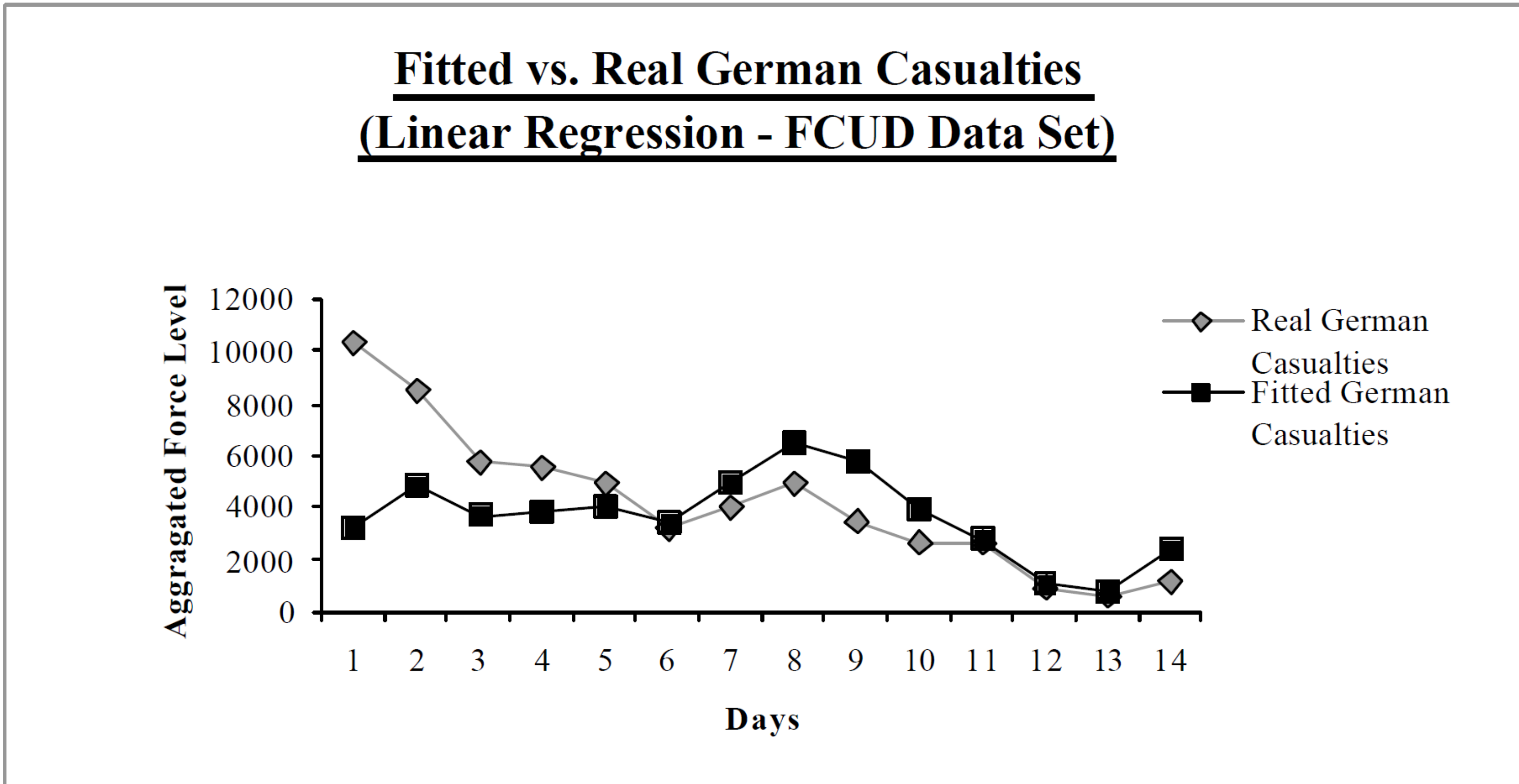



Figure III.5. Fitted versus real German casualties for FCUD data set. Except for the first two days, the fit appears much better than the CCUD analysis.

# Tracing the Battle for FCUD

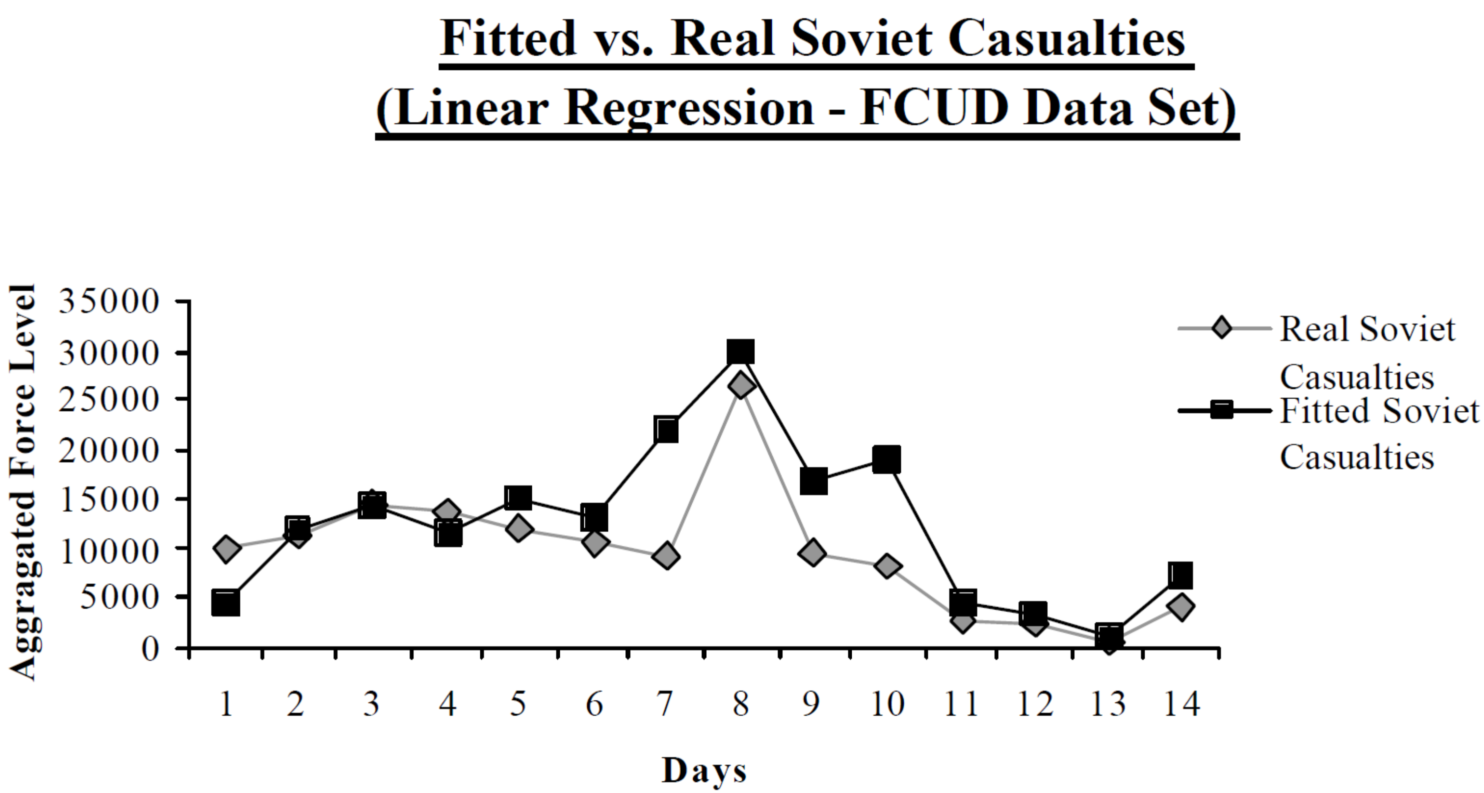



Figure III.6. Fitted versus real Soviet casualties for FCUD data set. Notice how the model's fitted casualties now closely resemble the real casualties. Most significantly, the model more accurately estimates the casualties from day eight. This greatly decreases the *SSR* and increases $R^2$.

# Breaking the Battle into Phases

- The first phase consists of the first two days of the campaign, with the Germans primarily attacking prepared defenses. The second phase, with the Germans pressing the attack against less-prepared defenses, contains days three through seven. The historic eighth day is unique and is considered a phase by itself. Of course, since this phase is only a single day, there is a perfect fit (i.e., no residual error); thus, this removes the (outlying) eighth day from the fits. The fourth and last phase is days nine through 14.

- The four-phase model fits are much better than what is obtained with the constant attrition coefficient models—for all three data sets. For both the ACUD and CCUD data, all of the basic Lanchester laws achieve $R^2$ values of between 0.732 and 0.744. The $R^2$ that is obtained by using the mean loss in each phase, for each side, as the sides' estimated losses, is 0.734 and 0.730, respectively, for the ACUD and CCUD data.

- As before, we get better fits from the FCUD data, with the $R^2$ values from the basic Lanchester laws ranging from 0.836 to 0.860. The FCUD phase mean model has an $R^2$ of 0.796.

**For modelers, attrition coefficients need to be updated as the battle changes (DUH)!**

# Lanchester & Kursk: Conclusions

- **Much better fits are obtained with FCUD data**.
- The data sets provides no conclusive differentiation among the basic Lanchester models  (though **linear and logarithmic better than square**).
- Much more of the variation in casualties is explained by the **phases of the battle**.
- **"The failure to find a good-fitting Lanchester model suggests that it may be beneficial to look for new ways to model highly aggregated attrition."**
  - **Is there a better meta model form?**

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# Final Thoughts

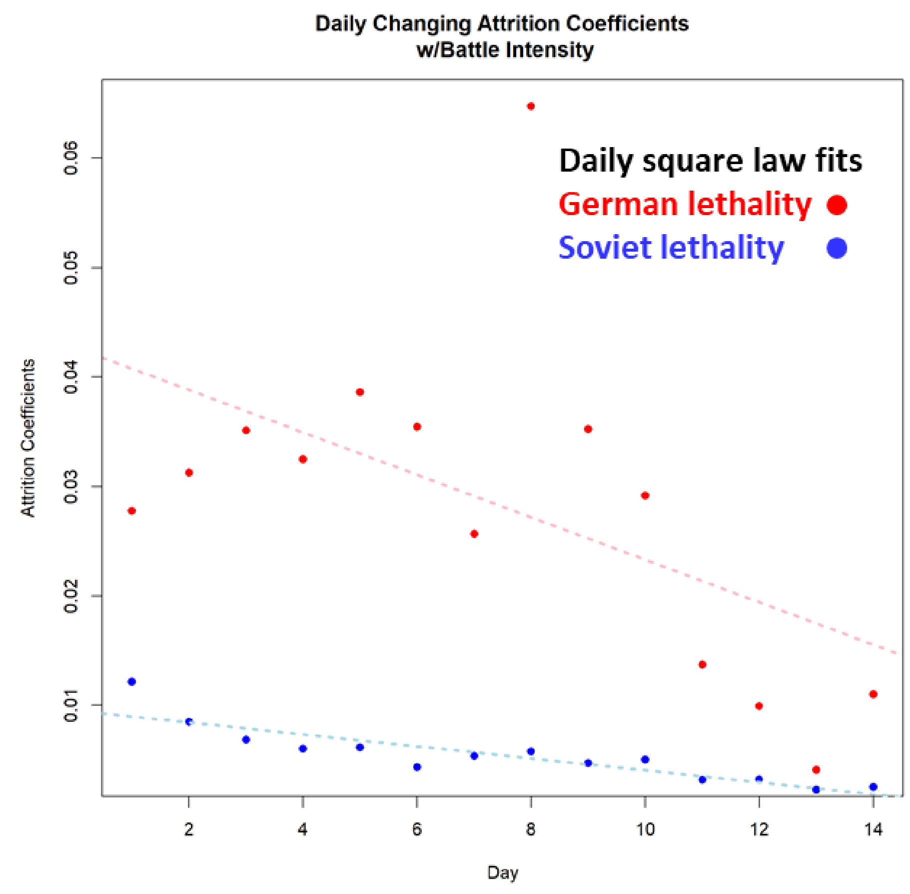
- **Each battle has its own unique circumstances…**

- **It is asking a lot to attempt to model combat attrition with its myriad of dynamic factors in time and space by simple paired differential equations.**

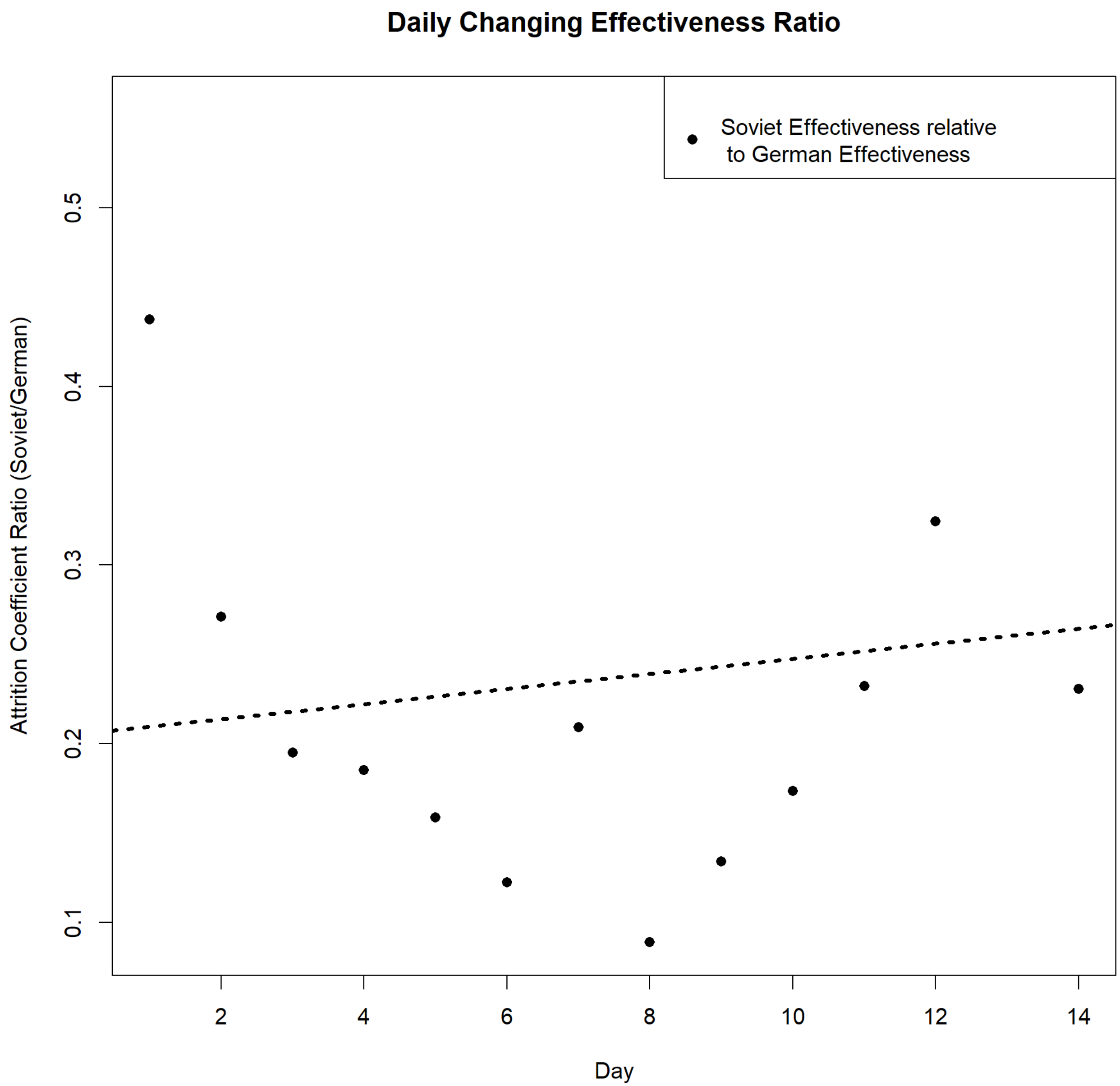
  - **"These differential equations, in order to be soluble, will have to represent extremely simplified forms of warfare; and therefore their range of applicability will be small." — Morse and Kimball**

  - **Constant attrition coefficients?**

    **"The Lanchester coefficients are often referred to as constants and it is easy to forget that this means only that the coefficients are assumed to be constant for a given battle or portion of a campaign." – Hartley and Helmbold**

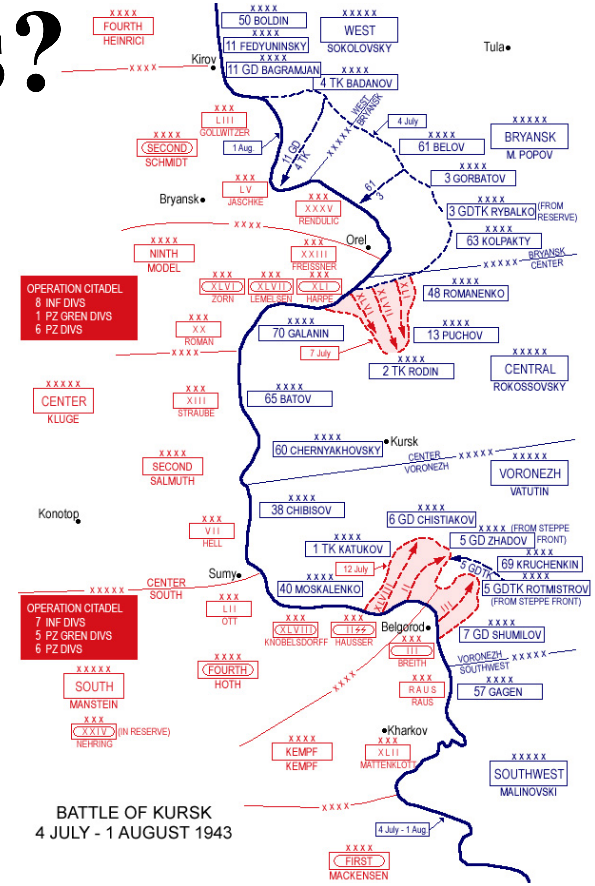


# Effectiveness Ratio (Soviet/German) by Day




Definitely not onstant and with clear trends

# Questions or Comments?

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